The Baltic Seminar Notes #6

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We continue on our discussion about well-foundedness of iterations and realizability. As for now, we have already introduced two kind of embedding argument:

Method 1. Suppose M is countable. Then by countably completeness, one can realize $Ult(M_0; F)$ into W_0 as the following diagram shows:

$$\begin{tikzcd}
W_0 & \\
M_0 \arrow[u, "\pi"] \arrow[r, "j_F"'] & Ult(M_0; F) \arrow[ul, bend right, "\sigma"']
\end{tikzcd}$$

where we can find a fiber δ by the countably completeness of $\sigma(F)$ and define $\sigma([a, f]) = \pi(f)(\delta\text{"}a)$.

Method 2. We again suppose that M is countable and $W_0^\omega \subset W_0$. We can thus copy $Ult(M_0; F)$ into $Ult(W_0; G)$ as the following diagram shows:

$$\begin{tikzcd}
W_0 \arrow[r, "j_G"] & Ult(W_0; G) \\
M_0 \arrow[u, "\pi"] \arrow[ur, dashed, "j_G \circ \pi"] \arrow[r, "j_F"'] & Ult(M_0; F) \arrow[u, "\sigma"']
\end{tikzcd}$$

Here, we define $\sigma([a, f]) = [\pi(a), \pi(f)]$ and it is thus elementary. By our assumption, $(M_0; \pi_0) \in W_0$. Notice that σ may not be in $Ult(W_0; G)$, since it may not be closed under ω-sequences. However, the existence of such σ is absolute among models with enough ordinal height.

Therefore $M_0, Ult(W_0; G)$ and $Ult(M_0; F)$ make up a realizing diagram. We can call this situation as "$Ult(W_0; G) \models F$ is $j_G \circ \pi_0$-realizable". Modulo j_G, we can now say that "$W_0 \models F$ is π-realizable".

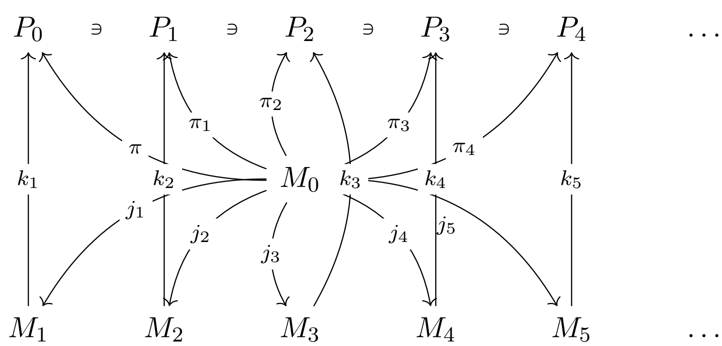
In a more general realizing argument, finding a realizing embedding σ can be complicated:

$$\begin{tikzcd}
P & \\
M \arrow[u, "\pi"] \arrow[r, "j_F"'] & Ult(M; F) \arrow[ul, dashed, bend right, "\sigma"']
\end{tikzcd}$$

where M and $Ult(M;F)$ are countable models but may not be inside P. To find such a σ, one only needs to find the images of functions $f : {}^{|a|}crit(F) \to M$ and the images of generators $a \in [\lambda]^{<\omega}$. Although the first task is easy since by the commutativity of the diagram, we have no choice but to set $\pi_F(f) \mapsto \pi(f)$, finding the images of generators to make σ elementary may not be easy and the absoluteness argument showed above may not be applicable in some situation. This is one of the key problem in inner model theory.

Theorem 1. *There is no iteration $\mathcal{I}$ on V such that $lh(\mathcal{I}) = \omega$ and $\mathcal{I}$ has no branch of length 2.*

Proof. Suppose $\mathcal{I}$ is such an iteration. Fix Θ big enough such that $\mathcal{I}$ happens on V_Θ. We take M as a countable hull of V_Θ, $\pi : M \to V_\Theta$ as the elementary embedding and consider $\mathcal{T} = \langle M_i, E_i : i < \omega\rangle$ as the required iteation happens on M:
Set $\kappa_i = crit(E_i)$ and $\lambda_i = lh(E_i)$. We now claim that $j_0(\kappa_0) > \kappa_1$. This is because E_1 is applied to M_0, $P^{M_1}(\kappa_1) = P^{M_0}(\kappa_1)$ and $P^{M_0}(\lambda_0) \supsetneq P^{M_1}(\lambda_0)$ since $E_0 \notin M_1$. This gives that $\kappa_1 < \lambda_0$ and thus $\kappa_1 < j_0(\kappa_0)$.



We now realize M_i into V_Θ by k_i. Since the diagram commutes, $\pi(\kappa_0) = k(j_0(\kappa_0)) > k(\kappa_1)$. We take $\tau_1 : P_1 \to P_0 = V_\Theta$ as the countably closed Skolem hull with size $k_1(\kappa_1)$[*]. Thus $P_1 \in P_0$. Keep repeating this construction, we construct a ill-founded sequence $\langle P_i \mid i \in \omega\rangle$ inside V. Contradiction. $\square$

Comment. Applying the same strategy, we can show that an iteration with length ω cannot be infinitely branching.

More on iteration trees: We have introduced the concept of normal iteration last time:

Definition. An iteration $\mathcal{I}$ is called normal if the following statements hold:

a. Each extender is applied to the least model it can be applied;

b. The length of extender grows.

Comment. Suppose E is a (κ, λ)-extender and F is a (κ', λ')-extender with $\kappa > \lambda'$. Then:

[*]Why we need to construct P_i for every stage? Well, if we keep realizing M_i in V_Θ, then $\pi(\kappa_1) > \pi(\kappa_0)$ and thus we can no longer compare $k_1(\kappa_1)$ and $k_2(\kappa_2)$.

- $V \to W_1 = Ult(V;F) \to W_2 = Ult(W_1; j_F(E))$ is a normal iteration;
- $V \to W_1 = Ult(V;E) \to W_2 = Ult(W_1;F)$ is not a normal iteration, since it violates clause a;
- $V \to W_1 = Ult(V;E)$, $V \to W_2 = Ult(V;F)$ is not a normal iteration, since it violates clause b;
- If extenders E and F are used along the same branch of a normal iteration with E used earlier, then $lh(E) < crit(F)$.

Theorem 2. *Suppose $\mathcal{I}$ is an iteration. Then for all $\alpha < \beta$, $M_\alpha \sim_{\lambda_\alpha} M_\beta$ and $M_\alpha \not\sim_{\lambda_\alpha+1} M_\beta$.*

Proof. We prove this by induction. To begin with, recall that we restrict every E_α to satisfy $strength(E_\alpha) = lh(E_\alpha) = \lambda_\alpha$ and λ_α is always inaccessible. Thus, $M_0 \sim_{\lambda_0} M_1$ and $M_0 \not\sim_{\lambda_0+1} M_1$ since $E_0 \notin M_1 = Ult(M_0;E_0)$. We now aim to show that $M_\alpha \sim_{\lambda_\alpha} M_{\alpha+1}$ and $M_\alpha \not\sim_{\lambda_\alpha+1} M_{\alpha+1}$.
Let $\beta = pred_T(\alpha+1)$ be the predecessor of $\alpha+1$ with respect to $<_T$. We compare M_α and M_β:

- Since we can apply E_α to M_β, we must have $P^{M_\alpha}(\kappa_\alpha) = P^{M_\beta}(\kappa_\alpha)$ and because of that, $Ult(M_\alpha;E_\alpha) \sim_{j_\alpha(\kappa_\alpha)+1} Ult(M_\beta;E_\alpha)$ where $j_\alpha : M_\beta \to M_{\alpha+1} = Ult(M_\beta;E_\alpha)$. To show the second statement, notice that every $f : [\kappa]^n \to V^M_{\kappa+1}$ can be coded by an element of $V^M_{\kappa+1}$. Since $P^{M_\alpha}(\kappa_\alpha) = P^{M_\beta}(\kappa_\alpha)$ gives $V^{M_\alpha}_{\kappa_\alpha+1} = V^{M_\beta}_{\kappa_\alpha+1}$ †, we have the required statement.
- As the definition of strength, we have $Ult(M_\alpha;E_\alpha) \sim_{\lambda_\alpha} M_\alpha$ and $Ult(M_\alpha;E_\alpha) \not\sim_{\lambda_\alpha+1} M_\alpha$.

By the above analysis, we have

$$M_{\alpha+1} = Ult(M_\beta;E_\alpha) \sim_{j_\alpha(\kappa_\alpha)+1} Ult(M_\alpha;E_\alpha) \sim_{\lambda_\alpha} M_\alpha;$$

$$M_{\alpha+1} = Ult(M_\beta;E_\alpha) \sim_{j_\alpha(\kappa_\alpha)+1} Ult(M_\alpha;E_\alpha) \not\sim_{\lambda_\alpha+1} M_\alpha.$$

Recall that $\lambda_\alpha \leqslant j_\alpha(\kappa_\alpha)$, we get $M_{\alpha+1} \sim_{\lambda_\alpha} M_\alpha$ and $M_{\alpha+1} \not\sim_{\lambda_\alpha+1} M_\alpha$
Now suppose τ is a limit ordinal and F is an extender used in $\mathcal{I}$ between stage $\gamma < \tau$. Then $crit(F) \geqslant \lambda_\gamma$ by the definition of normality of $\mathcal{I}$. Thus by our former argument, $M_\tau \sim_{\lambda_\gamma} M_\gamma$ and $M_\tau \not\sim_{\lambda_\gamma+1} M_\gamma$. Thus we have completed the proof. $\square$

Suppose $\mathcal{I}$ is a iteration and $lh(\mathcal{I})$ is a limit ordinal. Set $M(\mathcal{I}) = \bigcup_{\alpha<lh(\mathcal{I})} V^{M_\alpha}_{\lambda_\alpha}$. Suppose b is a cofinal branch in $\mathcal{I}$. Then $M(\mathcal{I}) = V^{M^{\mathcal{I}}_b}_{\delta(\mathcal{I})}$, where $V^{M^{\mathcal{I}}_b}$ is the direct limit of the models along the branch, and $\delta(\mathcal{I}) = \sup_{\alpha<lh(\mathcal{I})} \lambda_\alpha$. This can be easily shown by the theorem above.

†There is a way of coding H_{λ^+} into $P(\lambda)$, and $H_{\lambda^+} \supseteq V_{\lambda+1}$ if λ is inaccessible. See [2], Lemma 3.2.1., or [1], 2.2.

Theorem 3. *Zipper argument If b, c are two different cofinal branch of $\mathcal{I}$, then $\delta(\mathcal{I})$ is Woodin relative to functions in $M_b^{\mathcal{I}} \cap M_c^{\mathcal{I}}$ as witnessed by extender in $M(\mathcal{I})$.*

Thus, if there is no inner model with a Woodin cardinal, then every iteration tree would have unique well-founded cofinal branch, because otherwise $L(M(\mathcal{I})) \models$ "$\delta(\mathcal{I})$ is Woodin". We shall prove this in our next lecture.

References

[1] James Cummings. Iterated forcing and elementary embeddings. In *Handbook of set theory*, pages 775–883. Springer, 2010.

[2] Ronald Jensen. Manuscript on fine structure, inner model theory, and the core model below one woodin cardinal. `https://www.mathematik.hu-berlin.de/~raesch/org/jensen/pdf/book_skript_feb_11_2020.pdf`. Accessed: Mar. 1st, 2021.