The Baltic Seminar Notes #5

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Last time we showed the procedure of constructing ultrapower with a possibly outside extender, and showed that this ultrapower is well-founded. We now summarize some basic properties of extender ultrapower

Lemma 1. *Given that E is a (κ, λ)-extender in V, $Ult(V;E)$ satisfies the following properties:*

1. *$crit(j_E) = \kappa$ and $j_E(\kappa) > \lambda$;*
2. *Let $\alpha \leqslant \kappa$. If ${}^{\alpha}\lambda \subset Ult(V;E)$, then $Ult(V;E)$ is closed under α-sequences;**
3. *$E \notin Ult(V;E)$.*

Proof.

1. Recall that $j_E(\alpha) = [\{\kappa\}, c_\alpha]$. If $\alpha < \kappa$, then by Los' lemma and κ-completeness of the ultrafilter $E_{\{\kappa\}}$, $j_E(\alpha) = \alpha$ for all $\alpha < \kappa$. We claim that $[\{\kappa\}, \mathbb{U}] = \kappa$ in $Ult(V;E)$, where $\mathbb{U} : x \mapsto \bigcup x$.
2. Let $[a_\xi, f_\xi] \in Ult(V;E)$ for all $\xi < \alpha$. Since
$$j_E(\langle f_\xi : \xi < \alpha\rangle) = \langle j_E(f_\xi) : \xi < \alpha\rangle \restriction \alpha,$$
we only need $(a_\xi : \xi < \alpha) \in Ult(V;E)$ to have $\langle j_E(f_\xi)(a_\xi) : \xi < \alpha\rangle \in Ult(V;E)$.
3. Suppose $E \in M$, then we can define a surjection $e : {}^{\kappa}\kappa \to j_E(\kappa)$ in $Ult(V;E)$ by $e(f) = [\{\kappa\}, f]$. This shows $Ult(V;E) \vDash j(\kappa) \leqslant 2^\kappa$, which contradicts with the inaccessibility of $j(\kappa)$ in $Ult(V;E)$.

□

Remark. Suppose $\kappa < \kappa'$ are measurable cardinals with normal measures μ on κ and μ' on κ'. Then

a. Let E be the (κ, κ^+)-extender derived from the canonical ultrapower embedding $j_\mu : V \to Ult(V;\mu)$. Then $Ult(V;E) = Ult(V;\mu)$ and $j_E = j_\mu$;

b. Suppose $\kappa_0 < \kappa_1$ such that there is a (κ_0, λ_0)-extender E such that

• $\lambda_0 > \kappa_1$; • $Ult(V;E) \vDash$ κ_1 is strong; • $V_{\lambda_0} \subseteq Ult(V;E)$.

Let F be a (κ_1, λ_1)-extender in $Ult(V;E)$ such that $\lambda_1 > \lambda_0$. Then we can construct $Ult(V;F)$ and it is also well-founded.

*It is not hard to see that $Ult(V;E)$ is not always closed under ω-sequences.

Definition. We call κ a λ-strong cardinal iff there is an elementary embedding $j : V \to M$ such that:

$$\bullet\ crit(j) = \kappa; \quad \bullet\ j(\kappa) > \lambda; \quad \bullet\ V \sim_\lambda M.$$

If κ is λ-strong for all λ, then κ is called a strong cardinal.

Lemma 2. *Let E be a (κ, λ)-extender. Then the canonical ultrapower elementary embedding j_E satisfies:*

$$\bullet\ crit(j_E) = \kappa; \quad \bullet\ j_E(\kappa) > \lambda; \quad \bullet\ V \sim_\lambda Ult(V; E).$$

Thus, κ is λ-strong iff there is a (κ, λ)-extender.

We now state the procedure of iteration, sometime called an iteration tree.

Definition. Let M be some model of set theory(may not be full ZFC). We say $\mathcal{I} = (T, M_\alpha, E_\alpha : \alpha < \eta)$ is an iteration of M iff:

- $T = (\eta, <_T)$ is a tree on η, which means:

  $- <_T$ is a tree order on η; $\quad - \alpha <_T \beta \implies \alpha < \beta$; $\quad - pd_T(\alpha) = \{\beta < \eta : \beta <_T \alpha\}$ is a club of α.

- $E_\alpha \in M_\alpha$ and $M_\alpha \models E_\alpha$ is a countably complete $(\kappa_\alpha, \lambda_\alpha)$-extender.
- $M_0 = M$.
- $M_{\alpha+1} = Ult(M_\beta, E_\alpha)$ where β is the $<_T$ predecessor of $\alpha + 1$.

  With the above definition, one can compose all the intermediate embeddings between β and α given that $\beta <_T \alpha$. We denote this embedding by $j^{\mathcal{I}}_{\beta\alpha}$.

- If $\lambda < \eta$ is a limit ordinal, then $M_\lambda = \text{dirlim}(M_\beta, j^{\mathcal{I}}_{\beta\alpha} : \beta <_T \alpha <_T \lambda)$.

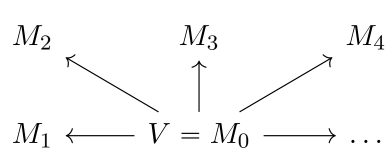
Remark. Some examples of iteration:

- Linear iteration is the easiest iteration:

  $$V = M_0 \xrightarrow{j_{01}} M_1 \xrightarrow{j_{12}} M_2 \xrightarrow{j_{23}} \dots \xrightarrow{j_{n\omega}} M_\omega \xrightarrow{j_{\omega,\omega+1}} M_{\omega+1} \xrightarrow{j_{\omega_1,\omega+2}} \dots \xrightarrow{j_{\alpha\infty}} M_\infty$$

  Here we take some extender from our last model and try to build the ultrapower with it, though it may not be the same extender every time.

- The iteration below is impossible:



I.e., a tree with infinite order must have an infinite branch. We shall explain the reason next time.

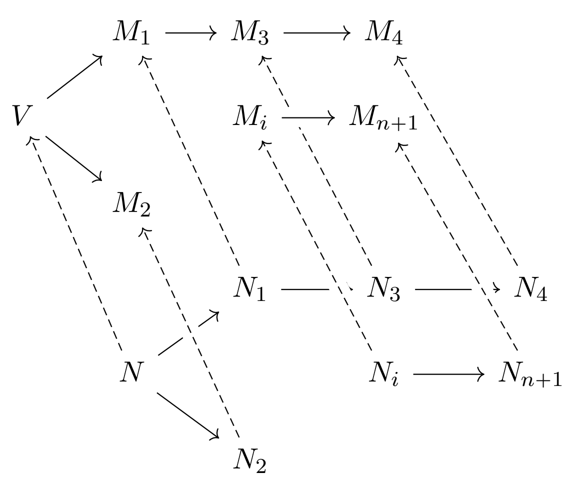
Definition. We say an iteration $\mathcal{I}$ is normal iff:

- For all $\alpha < \beta$, $\lambda_\alpha < \lambda_\beta$;
- For all α, let β be the predecessor of $\alpha + 1$. Then β is the least ordinal such that $\kappa_\alpha < \lambda_\beta$. †
- There is no overlapping extender. I.e., if $\alpha <_T \beta < \eta$ then $\kappa_\beta > \lambda_\alpha$.

Comment. The tree structure in the iteration is not removeable when considering very large cardinals. As an example, Foreman-Magidor-Shelah proved that a supercompact cardinal implies that there would be no projective well-ordering of reals. However, a linear iteration is well-founded is a Π^1_1-condition of reals, thus there would be a Δ^1_3 well-ordering of reals‡, which contradicts the existence of the supercompact. Therefore, considering iterations with higher complexity would be generally useful for very large cardinals.

Theorem 3. *Suppose $\mathcal{I}$ is a finite iteration of length n. Let $E_n \in M_n$ and let $M_{n+1} = Ult(M_i; E_n)$ where $i < n$ is the least such that $P(\kappa_n) \cap M_n = P(\kappa_n) \cap M_i$. Then M_{n+1} is well-founded.*

Proof. (Sketched) One can in fact collapse the whole picture down to a countable size and consider the following diagram:



Here, solid lines indicate the ultrapower embedding relation between models and dashed lines indicate the pull-back embeddings and embeddings realized by copying procedure. With an argument of proving well-foundedness, we can show that the countable completness of each E_n implies the well-foundedness of N_{n+1}, so M_{n+1} is well-founded. □

†Or: $\left(H_{\kappa_\alpha^+}\right)^{M_\beta} = \left(H_{\kappa_\alpha^+}\right)^{M_\alpha}$.

‡Idea: Let $x, y \in \mathbb{R}$ and define $x < y$ iff there exists some M such that M is iterable, x, y are both in some iteration of M and $x < y$ in that iteration.