The Baltic Seminar Notes #2

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Question: Are large cardinals first order properties?

Definition. κ is called λ-strong iff there exists some elementary embedding $j : V \to M$ such that $j(\kappa) > \lambda$ and $V_\lambda \subset M$. κ is strong iff it is λ-strong for every $\lambda \in Ord$.

Definition (Derived Ultrafilters)**.** Suppose we have $j : M \to N$ elementary* with M, N transitive, $M \models$ ZFC. The least ordinal moved by j is called the critical point, denoted as $crit(j)$. Set

$$U_j = \{X \in P(\kappa)^M \mid \kappa \in j(X)\}.$$

Fact. U_j is an M-ultrafilter. I.e., U_j is non-principal, κ-complete, normal and weakly amenable:

- (Non-principality) $\bigcap U_j = \varnothing$.
- (κ-completeness) For $\{X_i : i < \alpha < \kappa\} \in P(U_j) \cap M$, $\bigcap_{i<\alpha} X_i \in U_j$.
- (Normality) For every function $f : \kappa \to Ord$ which is regressive(i.e., $f(\alpha) < \alpha$) on a set $X \in U_j$, there is a $Y \in U_j$ s.t. f is constant on Y.
- (Weakly Amenability) For every $F \in {}^\kappa M \cap M$, $\{\xi < \kappa : F(\xi) \in U\} \in M$.

Remark. Weakly amenability generally says that for every size κ set of M, its intersection with U_j is in M, which is useful for defining the corresponding ultrafilter in the iteration. U_j would not be amenable(That is, $U_j \cap A \in M$ iff $A \in M$) in general, since if $U_j \notin M$, then $P(\kappa)^M \cap U_j = U_j \notin M$.

We now state a procedure of constructing the iterated ultrapowers. For the first stage, suppose U is an M-ultrafilter. Set a relation $\equiv_U$ in $f, g \in {}^\kappa M \cap M$ with:

$$f \equiv_U g \iff \{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U.$$

Let $[f]$ denote the equivalent class† with respect to f. Let $D = \{[f] \mid f \in {}^\kappa M \cap M\}$. We then define the membership relation ϵ over D as:

$$[f]\epsilon[g] \iff \{\alpha : f(\alpha) \in g(\alpha)\} \in U.$$

Then the ultrapower $Ult(M; U) = (D; \epsilon)$. If the relation ϵ is well-founded, we recognize the ultrapower as its transitive collapse. Moreover, it is easy to check that the following mapping is an elementary embedding:

$$j_U : M \to Ult(M; U), \quad x \to [c_x];$$

where c_x is the constant sequence with value x. j_U is often called the canonical embedding.

*Elementary embeddings are always supposed to be non-tivial, i.e., $j \neq id$.

†To avoid the case that $[f]$ being a proper class, we can use a strategy called the Scott's trick, namely, identify $[f]$ with only those members with minimal rank.

Fact. Ultrapower constructed by a derived ultrafilter is well-founded.

Proof. Let $j : M \to N$ be the elementary embedding and $crit(j) = \kappa$. Then we can define an elementary embedding $k : Ult(M; U_j) \to N$ as:

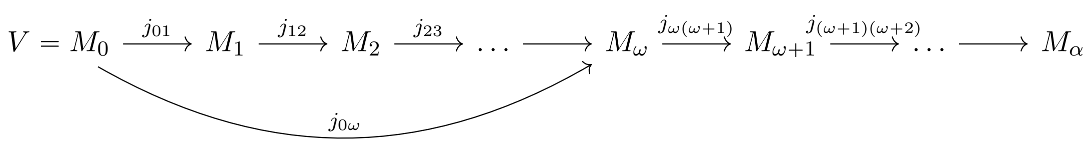
$$k([f]) = j(f)(\kappa).$$

□

To iterate the construction of ultrapower, suppose $U \in V$ is an ultrafilter over κ. Denote $M_1 = Ult(V; U)$, $j_{01} = j_U$ and $U_1 = j_{01}(U_1)$. Similarly, we can then define $M_n, j_{n(n+1)}$ and U_n. To define the ωth stage, we use the direct limit and define M_ω as the direct limit[‡] of the direct system $(M_n, j_{nm} \mid n < m < \omega)$, where $j_{nm} = j_{(m-1)m} \circ ... \circ j_{n(n+1)}$. We also define $j_{n\omega} M_n \to M_\omega$ as the natural embedding induced by the direct limit. It is then obvious that this procedure can be iterated through all ordinals.

Definition. We say $(M_\beta, \pi_{\beta\gamma} : \beta < \gamma < \alpha)$ is the αth putative iteration of $(V; U)$ if:

- $M_0 = V$;
- $j_{\beta\gamma} : M_\beta \to M_\gamma$ is elementary for $\beta < \gamma$;
- $M_{\beta+1} = Ult(M_\beta; U_\beta)$; $U_\beta = j_{0\beta}(U)$;
- $M_\lambda = \text{dirlim}(M_\beta, j_{\beta\gamma} : \beta < \gamma < \lambda)$ if λ is a limit ordinal.



Comment. [§] We can define the length-α putative iteration of any $(M; U)$ even if $\alpha \geqslant M \cap Ord$.

Definition. We say $(M; U)$ is α-iterable if the model of length-α putative iteration of $(M; U)$ is well-founded. If $(M; U)$ is α-iterable then we call $(M; U)$ iterable.

Theorem 1 (Gaifman)**.** *Suppose $V \models$ "U is ω_1-complete. Then $(V; U)$ is iterable.*

Remark. There is an easier proof of the above theorem which generally says: If it is not α-iterated, then by countably completeness, one can take $\bigcap_{n \in \omega} \{s \in \rho^\alpha \mid f(n) \ni f(n+1)\}$ which is non-empty. Thus the ground model is ill-founded. However this requires the iterated ultrapower to be defined in an old-school manner. Readers interesting in this construction can refer to [3], Chapter 3 and [2], Chapter 19.

[‡]For more information of direct limit, see [1], Lemma 12.2.

[§]Remarks are always due to the note taker and comments are due to the lecturer or the audience.

Somewhat equivalently, we can also prove the above theorem by choosing the minimal pair (γ, ξ) that $M_\gamma \models$ "Ordinal below $j_{0\gamma}(\xi)$ is ill-founded". Let the ill-founded sequence be $(x_i \mid i < \omega)$. We can then choose the minimal α with $M_\alpha \models$ "$j_{0\alpha}(\xi) > \nu$" where $j_{\alpha,\gamma}(\nu) = x_0$. With these minimalities, M_α sees that in M_γ, ordinals below $j_{\alpha\gamma}(\nu)$ is well-founded, contradiction.

The above proof is due to Gaifman. The note taker generally believes that the above two proofs are saying the same thing, but the requirement of these two proofs are slightly different:

- Gaifman's proof works in V and requires a quite natural result, that is, the factor lemma([1], Lemma 19.5).
- Kunen's proof works even if $U \notin V$. Also, it provides a more direct way to grasp the iteration by only constructing one ultrafilter.

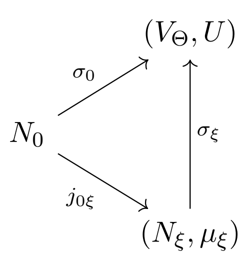
Moreover, countably completeness is not a necessity for iterability. See [3] and [2]. Also, this theorem has some considerable generalizations, like doing ultrapower outside of the model, taking extender ultrapowers, or taking ultrapower from a set of ultrafilters or extenders.

Proof. Our proof for this theorem is due to Jensen. Let α be such that $(V; U)$ is not α-iterable, and $U \in V_\Theta$ for a sufficiently big Θ. Let $\lambda > \Theta > \alpha$ be such that $V_\lambda \models$ "$(V_\theta; U)$ is not α-iterable". Let $\sigma : M \to V_\lambda$ be elementary and M be countable, such that $U \in \text{ran}(\sigma)$ and $\alpha \in \text{ran}(\sigma)$. Let $\mu = \sigma^{-1}(U), \beta = \sigma^{-1}(\alpha), N = \sigma^{-1}(V_\Theta)$.

The above assumptions imply that $M \models$ "M is not β-iterable", and if $N = V^M_{\bar{\Theta}}$, where $\bar{\Theta} = \sigma^{-1}(\Theta)$, then $M \models$ "N is not β-iterable".

Let $(N_\xi, i_{\xi\gamma} : \xi < \gamma < \beta)$ be the model of length-β putative iteration of (N, μ). We will define $\sigma_\xi : N_\xi \to V_\Theta$ such that:

- $\sigma_0 = \sigma \restriction N$;
- $\sigma_\xi = \sigma_\gamma \circ j_{\xi\gamma}$.



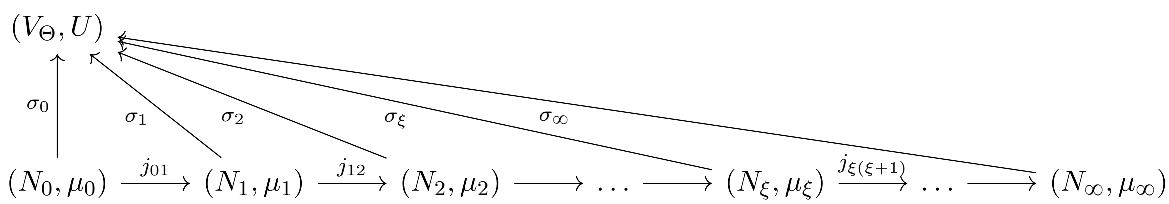
First Step Lemma: Suppose $\sigma : (N; W) \to (V_\Theta, U)$ where N is countable, then there is $k : Ult(N; W) \to V_\Theta$ such that $k \circ j_W$.(k is often called the realizability embedding).

Proof. We need to define $k([f]_W)$. Let $\delta \in \bigcap \sigma"W$: It is clear that $\bigcap \sigma"W$ is not empty, since $\sigma"W \subset U$, $\sigma"W$ is countable and U is countably complete. We then set $k([f]_W) = \sigma(f)(\delta)$. We claim that $k : Ult(N, W) \to V_\Theta$ is elementary.

To prove this claim, suppose $Ult(N, W) \models \phi([f])$ for some sentence ϕ. Then by Los' theorem, $A = \{\xi : W_\Theta \models [f(\xi)]\} \in W$; so $\delta \in \sigma(A)$. Since $\sigma(A) = \{\xi : V_\Theta \models \phi[\sigma(f)(\xi)]\}$, we have $V_\Theta \models \phi[\sigma(f)(\delta)]$, thus $V_\Theta \models \phi[k([f])]$. □

Let $\sigma_{\xi+1} : N_{\xi+1} \to V_\Theta$ and let k be defined above. We then construct $\sigma_\lambda : N_\lambda \to V_\Theta$ for a limit ordinal λ. If $X \in N_\lambda$, then fix $\gamma < \lambda$ and $y \in N_\gamma$ such that $X = j_{\gamma\lambda}(y)$. Set $\sigma_\lambda(x) = \sigma_\gamma(y)$. It is left to the reader to show that σ_λ is well-defined and elementary. Thus, we proved that (N, W) is countably iterable and at each stage, the iteration is embeddable into (V_Θ, U). The well-foundedness of V then leads to a contradiction. □

Remark. There is some considerable pros and cons in this narrative, but this proof is also written on Steel's notes [4] and [2], Theorem 19.11. However Kanamori only proved a half of this theorem, namely if U is countably complete, then the pair $(M; U)$ is countably iterable.
The idea of this proof is that we can discuss the iteration in a countable toy model, and the countable completeness(in V) natrually gives the countable iterability of the toy model, since at each stage, a δ is given, which can be regarded as a "critical point" to move on. The following figure is due to [4] and the note taker found it is helpful for developing the intuition of this proof.



Theorem 2. *Suppose there are proper class of measurable cardinals. Then Σ^1_3-generic absoluteness holds, i.e.: If ϕ is Σ^1_3 and G is (set) generic then:*

$$V \models \phi \iff V[G] \models \phi.$$

Proof. Let $\mathbb{P}$ be the forcing poset and let λ_0 be a big enough cardinal such that $\mathbb{P} \in V_{\lambda_0}$ and λ be a measurable above λ_0. Let U be a normal measure over λ.
$V \models \phi \implies V[G] \models \phi$ follows from the Shoenfields absoluteness theorem, see [2], Theorem 13.15. We now present the other direction. Let $x \in \mathbb{R}^{V[G]}, V[G] \models \phi[x]$ with ϕ being Σ^1_3 and $\phi = \exists u\psi[u]$. Let $\pi : M \to V_\Theta$ be elementary such that Θ is big enough and larger than λ. We also assume that $U \in \operatorname{ran}(\pi), \mathbb{P} \in \operatorname{ran}(\pi)$. Let $k = \pi^{-1}(\lambda), \mu = \pi^{-1}(U)$ and $\mathbb{Q} = \pi^{-1}(\mathbb{P})$.
It follows from the settings that $M \models \Vdash_{\mathbb{Q}} \exists u\psi(u)$, where ψ is a Π^1_2 formula. Let $H \in V$ be M-generic w.r.t. $\mathbb{Q}$, so $M[H] \models \exists u\psi[u]$. Let $y \in \mathbb{R}^{M[H]}$ be s.t. $M[H] \models \psi[y]$. We are now trying to construct the iteration to "stretch" the model to contain ω_1, so we can finally apply the Shoenfield's Absoluteness.
By the countably completeness of U, $(M; \mu)$ is α-iterable for all $\alpha < \omega_1$, and by [2], Lemma 19.12, the pair is iterable. Let $(M_\alpha : \alpha \leqslant \omega_1)$ be the length-ω_1 putative iteration, then since

λ is picked arbitrarily large, the forcing poset $\mathbb{P}$ is below it, so the elementary embedding $j_{0\omega_1} : M \to M_{\omega_1}$ can be lifted up to the forcing extension $j^+_{0\omega_1} : M[H] \to M_{\omega_1}[H]$ by simply setting

$$j^+(\tau_H) = j(\tau)_H.$$

It is easy to show that j^+ is elementary and $j^+ \restriction M = j$. Thus, by elementarity and $M[H] \models \psi[y]$, $M_{\omega_1}[H] \models \psi[y]$. Also, since $\omega_1 \subseteq M_{\omega_1}[H]$, we can apply Shoenfield's Absoluteness to get $V \models \psi[y]$. Thus $V \models \exists u \psi[u]$. □

References

[1] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[2] Akihiro Kanamori. *The higher infinite: large cardinals in set theory from their beginnings*. Springer Science & Business Media, 2008.

[3] Kenneth Kunen. Some applications of iterated ultrapowers in set theory. *Annals of Mathematical Logic*, 1(2):179–227, 1970.

[4] John Steel. An introduction to iterated ultrapowers. In *Forcing, Iterated Ultrapowers, And Turing Degrees*, pages 123–174. World Scientific, 2016.