The Baltic Seminar Notes #1

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Definition. Some notions and terms:

- Reals or real numbers are members in the Baire space ω^ω, where we can define a topology with basic open sets $N_s = \{x \in \omega^\omega : s \prec x\}$ for all $s \in \omega^{<\omega}$.
- For any infinite cardinal κ, we say $T \subset \bigcup_{n\in\omega} \omega^n \times \kappa^n$ is a tree if it is closed on initial segments. We then denote $[T]$ as the collection of all branches of T. When replacing $\omega^n \times \kappa^n$ with X^n, T is also called a tree on a set X.

Fact. Closed sets are of the form $[T]$ for some tree T.

Remark. *Proof.* ([1], 12.10) Consider $T = \{x \restriction m \mid x \in C \wedge m \in \omega\}$ for any closed set C. Use the definition of N_s and topology base. □

Definition. Axiom of determinacy(ommited).

Theorem 1. *Important theorems about determinacy:*

1. *(Martin) ZFC+$\exists\kappa$(κ is measurable) $\vdash$ Analytic Determinacy;*
2. *(Martin) ZFC $\vdash$ Borel Determinacy;*
3. *(Martin-Steel) ZFC+infinitly many Woodin cardinals $\vdash$ Projective determinacy;*
4. *(Martin-Steel-Woodin) ZFC+infinitly many Woodin cardinals+a measurable cardinal above them $\vdash$ $AD^{L(\mathbb{R})}$.*

We are going to prove the first one today and leave the last two as the aim of this seminar.

1 Shoenfield absoluteness

Lemma 2 (Shoenfield's tree)**.** *Given $\kappa \geqslant \omega$, there is a tree $T \in L$ on $\omega \times \kappa$ such that*

$$p[T] = \{x \in \omega^\omega : x \text{ codes a well-founded structure}\},$$

where $p[T]$ is the collection of the projection of every branch of T.

Remark. In case that we shall later use the exact construction of this tree, I wrote something below for myself to learn this absoluteness theorem. This is not contained in this lecture.

Definition. Let $WF \subset \omega^\omega$ be the set defined as:

$$x \in WF \iff E_x \text{ is well-founded and extensional.}$$

Here, $E_x \subset P(\omega^2)$ is defined as: $\langle m, n\rangle \in E_x$ iff $x(2^m 3^n) = 0$.

Natrually, we has defined a way of coding any countable and transitive $\in$-structure by some $x \in \omega^\omega$, and WF is thus the collection of these codes.

Fact ([1], 13.6)**.** $WF \in \Pi^1_1 - \mathbf{\Sigma}^1_1$.

We next prove the Π^1_1 version of the Shoenfield's Absoluteness theorem. Notice that this theorem can be generalized to $\mathbf{\Sigma}^1_2$. Thus by applying the theorem to WF, we complete the proof for Lemma 2.

Theorem 3 (Shoenfield, 1961; [1], 13.14)**.** *If $A \subset \omega^\omega$ is Π^1_1, there is a tree $T \in L$ on $\omega \times \omega_1$ such that $A = p[T]$.*

Proof. Using the tree representation of $\mathbf{\Pi}^1_1$ sets, we have: There is a tree U on $\omega \times \omega$ such that

$$x \in A \iff T_x \text{ is well-founded } \iff \exists g : T_x \to \omega_1 \text{ order preserving.}$$

Let $\langle s_i \mid i \in \omega \rangle$ be the canonical order of $\omega^{<\omega}$. Define the following tree on $\omega \times \omega_1$:

$$T = \{\langle s, u \rangle \mid \forall i, j < |s| (s_i \supset s_j \wedge \langle s \mid |s_i|, s_i \rangle \in T \implies u(i) < u(j))\}.$$

This is the tree we want. □

Remark. ω_1 can be replaced with any $\kappa > \omega$ here.

2 Homogeneous Suslin Sets

Let κ be an infinite cardinal and let $meas(\kappa)$ be the collection of all ω_1-complete ultrafilter on some $[\kappa]^n$.

A function $\mu : \omega^{<\omega} \to meas(\kappa)$ is called a homogeneous system iff:

- For all $s \in \omega^{<\omega}$, $\mu(s)$ concentrantes on $[\kappa]^{|s|}$;
- For all $s \prec t \in \omega^{<\omega}$, $\mu(t)$ projects to $\mu(s)$. I.e., for all $A \in \mu_t$, $p_{t,s}[A] = \{a \upharpoonright lh(s) : a \in A\} \in \mu_s$.

Definition. Ultrapower construction(ommited).

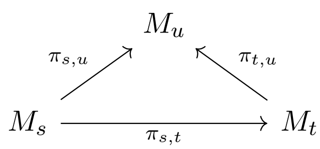
Suppose μ is a homogeneous system. We have $\forall s \in \omega^{<\omega}$, $Ult(V, \mu_s)$ is well-founded and $\pi_s : V \to Ult(V, \mu(s)) = M_s$ where $\pi_s = \pi_{\mu_s}$, the natrual embedding.

Fact. We have a map $\pi_{s,t} : M_s \to M_t$ for $s \prec t$ given by

$$\pi_{s,t}([f]_{\mu_s}) = [f']_{\mu_t}$$

where $f'(a) = f(a \upharpoonright lh(s))$. Then $\pi_{s,t}$ is an elementary embedding, and for $s \prec t \prec u$, we have

$$\pi_{s,u} = \pi_{t,u} \circ \pi_{s,t}.$$



Definition (Homogeneous Suslin)**.** We say $A \subset \omega^\omega$ is homogeneous Suslin iff for some homogeneous system $\mu : \omega^{<\omega} \to meas(\kappa)$,

$$x \in A \iff (\mu_{x \upharpoonright n} : n \in \omega) \text{ is well-founded(i.e., has well-founded direct limit).}$$

Lemma 4. *Suppose μ is a homogeneous system and $x \in \omega^\omega$. Then the following statements are equivalent:*

a. *$M_x = \text{dirlim}_{n \to \infty} Ult(V, \mu_{x \upharpoonright n})$ is well-founded;*

b. *For all $(A_i : i < \omega)$ such that $A_i \in \mu_{x \upharpoonright i}$, there is a fiber through them, i.e., there exists $f : \omega \to \kappa$ such that for all $i \in \omega$, $f \upharpoonright i \in A_i$.*

Proof. a implies b: Suppose M_x well-founded and $(A_n : n \in \omega)$ is such that $A_n \in \mu_{x \upharpoonright n}$. Consider the relation R whose branches are exactly fibers there. Namely, $\text{dom}(R) = \bigcup_{n \in \omega} A_n$ and

$$bRa \iff a \prec b \wedge b \in A_{|b|}.$$

We claim that R is ill-founded.

Remark. A such ill-founded sequence can be constructed by the projecting property and countable completeness of $\mu_{x \upharpoonright n}$.

b implies a: Suppose M_x is ill-founded. Then by the fiber granted by b, the ill-foundedness can be considered in V. □

3 Analytic Determinacy

Theorem 5 (Martin)**.** *Suppose A is homogeneous Suslin. Then G_A is determined.*

Fact. Suppose A is homogeneous Suslin. Let $\mu : \omega^{<\omega} \to meas(\kappa)$ be a homogeneous system s.t. $x \in A \iff M_x$ is well-founded. Then there exists a tree T on $\omega \times \kappa$, s.t.

1. $A = p[T]$
2. For all $s \in \omega^{<\omega}$, $T_s \in \mu_s$ where $T_s = \{u : (s, u) \in T\}$ *;

Proof. Suppose μ is a homogeneous representation of A and T is as above. We consider a new game G_A^*:

*Also called "T is homogeneous".

I	(n_0, α_0)	(n_2, α_2)	$\ldots$
II	n_1	n_3	$\ldots$

Player I wins iff $(x, f) \in [T]$, where $x = (n_0, n_1, ...)$ and $f = (\alpha_0, \alpha_2, ...)$. Otherwise Player II wins. We claim that G^*_A is a closed game; i.e., if Player I loses the game, then he already loses it at a finite stage. We also claim that every closed game is determined.

Assume Player I has a winning strategy, then since the winning protocol of G^*_A is more strict than G_A for Player I, he can use the same strategy to defeat Player II.

Assume Player II has a winning strategy and let it be Σ^*. We now build a winning strategy Σ for Player II to win G_A. Let

- $\sigma(n_0) = n_1$ iff $\{\alpha_0 : \Sigma^*(n_0, \alpha_0) = n_1\} \in \mu_{\langle n_0 \rangle}$;
- $\sigma(n_0 n_1 n_2) = n_3$ iff $\{(\alpha_0, \alpha_2) : \Sigma^*((n_0, \alpha_0) n_1 (n_2, \alpha_2)) = n_3\} \in \mu_{\langle n_0 n_1 n_2 \rangle}$, etc.

We claim that σ is a winning strategy for Player II. Let

$$A_k = \{(a_0, ..., a_{2k}) : \Sigma^*((n_0, \alpha_0) n_1 ... (n_{2k}, a_{2k})) = n_{2k+1}\}.$$

We have $A_k \in \mu_{\langle n_0, ..., n_{2k} \rangle}$. We want to see that $x \notin A \iff M_x$ is ill-founded. Suppose M_x is well-founded. There is a sequence $\alpha = (\alpha_0, ...)$ s.t. for all $k \in \omega, \alpha \upharpoonright k \in A_k$. By the specific structure of T, $(x \upharpoonright n, (\alpha_0, ..., \alpha_{2n})) \in T$ for all n. But Player II wins the game G^*_A. Contradiction! □

Lemma 6. *$\exists \kappa$(κ is measurable) $\implies \Pi^1_1$ sets are homogeneous Suslin.*

Proof. (As a part of the proof of [1], 31.1) Let U be a normal ultrafilter over κ and for each $s \in \omega^{<\omega}$, define:

$$X \in U_s \iff X \subset X \subset T_s \wedge \exists H \in U(|H|^{|s|} \subset \text{ran}"X).$$

□

References

[1] Akihiro Kanamori. *The higher infinite: large cardinals in set theory from their beginnings.* Springer Science & Business Media, 2008.